# STABLE PROGRADE EARTH-MOON MULTI-ORBITER CYCLERS VIA THREE-BODY DYNAMICS

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**Abstract—** Periodic trajectories in the Circular Restricted Three Body Problem that have close encounters with both the primary and the secondary, also known as multi-orbiter cyclers, have been gaining interest due to their potential for use in cislunar missions. The objectives of the present study was to expand previous results on multi-body cyclers to systems with higher values of the mass parameter, and investigate the universal existence of stable subfamilies. Several cycler trajectories were discovered in binary star system mass parameter range, and substantial evidence was discovered in favor of the existence of a stable subfamily for every cycler family.

## INTRODUCTION

As interest in multi-body dynamics grows with interest in the expansion cislunar infrastructure for both commercial and military applications, interest in efficient methods for transportation in the Earth-Moon system grows proportionally. An evolving area of research which addresses the need for better access to Cislunar space is the study of periodic orbits in the Circular Restricted Three Body Problem (CR3BP), a simplified variation on the Three Body Problem from classical mechanics which governs the motion of a small body in the field of two much larger bodies known as the primary and the secondary masses. A *cycler* trajectory in the CR3BP is a periodic orbit which has close encounters with both the primary and secondary mass. These trajectories have significant potential not just spacecraft trajectory design, but celestial mechanics as well.

The objective was to develop the framework around multi-orbit cyclers. An open question was whether or not a stable subfamily exists for every family of multi-orbit cyclers. An additional objective was to find stable cyclers for larger values of $\mu$, in order to show that it is possible for a planet to exist in a cycler orbit in a binary star system. Finally, the method described in[7] was refined and a more formal method of locating cyclers was developed.

### Literature Review

The concept of a cycler orbit is not new, and has existed in the literature for several decades. In 1963, Arsentorf showed that periodic solutions that have close encounters with both bodies exist via analytic continuation of Keplarian orbits for small $\mu$, and demonstrated two numerical examples of these trajectories in the Earth-Moon system[2]. In 1964, Davidson provided several numerical orbits of cyclers labeled as "transition orbits"[3]. Szebehely featured cycler orbits in his text on the CR3BP using the terminology "alternating satellite"[8]. Aldrin coined the term "cycler" in a presentation in 1985 wherein he proposed the use of cyclers for several applications for international cislunar infrastructure development[1]. A drawback of Aldrin's proposal was that he considered only retrograde cycler orbits which require considerably more energy to achieve. Leiva and Briozzo submitted several publications that featured cycler orbits, such as[6],[5]. Wittal, Miaule, and Asher published an article on the use of cyclers orbits for the Artemis missions in 2022, but like Aldrin, considered only retrograde cyclers. The present study builds upon the framework, methods, and results presented in Ross and Roberts-Tsoukkas[7], which formalized many of the results obtained prior and showed that stable cycler subfamilies existed, which had not been known previously.

Technical Background

In this section, a brief overview of the mathematical model and some of its key structures properties, including invariant manifolds and

The Three Body Problem is one of the oldest and most studied problems in nonlinear dynamics. Given the position and velocities of three masses at some time $t_0$, each which exert a gravitational force according to Newton's Law of Gravitation, the problem is to determine the positions and velocities of each mass at an arbitrary time $t$, which constitutes a solution to the problem. It is well known that a closed-form solution does not exist to the problem, but several periodic solutions to the problem have been discovered using analytical and numerical approaches. A common variant of the Three Body Problem is the Circular Restricted Three Body Problem (CR3BP) which makes a few simplifying assumptions, namely, that two of the masses, called the *Primary* and *Secondary* respectively, are much larger than the third mass, and the Primary and Secondary orbit follow circular orbits around their common mass centers. The CR3BP has arguably been studied more than the full 3BP, due to its applications to spacecraft trajectory design in the Earth-Moon system, as well as some celestial mechanics applications, but also on account of its simplicity. Whereas one must determine three trajectories in the full 3BP, one needs only determine the trajectory of the third mass in the CR3BP, significantly reducing the dimensionality of the problem. We will also restrict to the planar case of the CR3BP with the notion that the results discussed presently extend in some way to the full spatial problem, but such an extension is outside of the scope of the present study.

The equations of motion for the CR3BP are commonly written in a reference frame that rotates with the axis passing through the primary and secondary and centered at their common center of mass, viz.

$$\begin{aligned} \ddot{x} - 2\dot{y} - x &= \frac{\partial U}{\partial x} \\ \ddot{y} + 2\dot{x} - y &= \frac{\partial U}{\partial y} \end{aligned} \tag{1}$$

where $(x, y)$ is the coordinate of the third mass with respect to the rotating frame, and $U$ is the gravitational potential function viz.

$$U(x,y) = -\frac{1-\mu}{\sqrt{(x+\mu)^2+y^2}} - \frac{\mu}{\sqrt{(x-1+\mu)^2+y^2}} \tag{2}$$

Note the notation $\dot{f} \equiv df/dt$, $\ddot{f} \equiv d^2f/dt^2$. The system of equations (1) has a conserved quantity known as the *Jacobi Constant* $C = C(x, y, \dot{x}, \dot{y})$ given by

$$C(x,y,\dot{x},\dot{y}) = -(\dot{x}^2+\dot{y}^2)+(x^2+y^2)-2U(x,y) \tag{3}$$

We omit, for brevity the details on the theory of invariant manifolds and periodic orbits in the CR3BP, but suggest Koon et al. 2022[4] for a more in-depth treatment. It is important that we label two Poincaré sections of interest, as depicted in Figure 1, as these will be used to classify Cyclers.

## METHODS

In this section, a brief exposition on the classification of three-body cycler orbits will be given, followed by a geometric method for locating cyclers of specific classifications. In addition, we will briefly cover single-parameter continuation methods that were used in the study.

### Classification of Cyclers

A $(k_1, k_2)$ cycler with $k_1, k_2 \in \mathbb{N}$ is a periodic solution to (1) that crosses the $U_1^-$ section $k_1$ times and $U_2^+$ $k_2$ times. A $(k_1, k_2)$

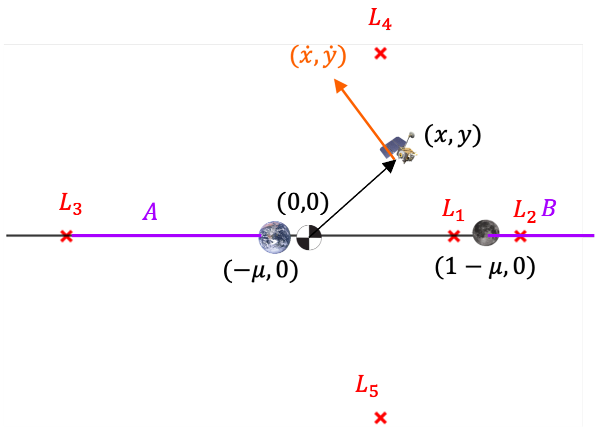


Figure 1: A schematic showing the planar circular restricted three body problem (PCR3BP model). The primary and secondary (in this case, the Earth and the Moon are plotted, alongside the 5 equilibria ($L_i$), marked with red x's. Additionally, the Poincaré sections $A$ and $B$ are plotted which will be used to define $(k_1, k_2)$ cyclers.

cycler is considered symmetric if it has two perpendicular crossings of the $x$-axis. A cycler is considered *stable* if locally it is Lyapunov stable, meaning that trajectories within some small neighborhood of the cycler remain nearby and do not diverge with time. Note that stability is determined only with respect to the planar problem and may not persist in the full spatial CR3BP.

Some Preliminaries

It has been shown[4] that regions of transport are partitioned by the invariant manifolds of the Lyapunov periodic family orbit. Moreover a trajectory that transits from the primary realm to the secondary realm lies "within" the stable manifold of the $L_1$ Lyapunov orbit, and a trajectory that came from the secondary realm to the primary realm must lie "within" the unstable manifold of the $L_1$ Lyapunov orbit. It follows that a cycler orbit, which periodically travels between the primary and secondary realm must lie within the intersection of the stable and unstable manifold of the $L_1$ Lyapunov orbit.

To ease the subsequent section, we will define two evolution operations $P_A$ and $P_B$ which evolve a state $X$ under the dynamics in (1) until the sets $A$ and $B$ are encountered respectively. Thus, it follows that $P_A(X) \in A$ and $P_B(X) \in B$ for all states $X$.

Locating Cyclers

Suppose a particular $(k_1, k_2)$ cycler is desired at some $\mu$ and at a Jacobi Constant $C_0$. For the sake of brevity, we will assume that both $k_1$ and $k_2$ are odd, but note that the method will still work if one or both $k_1$ and $k_2$ are even, albeit with some slight modifications. Construct a set $R$ of states $X = (x_0, 0, 0, \dot{y}_0)$ for which $C(x, 0, 0, \dot{y}) = C_0$ and that define transit trajectories in both forward and backward time. The set $R$ describes the set of candidate cyclers. Suppose $R \subseteq A$ was obtained from evolving the invariant manifolds of the $L_1$ Lyapunov orbit forward until $A$ was encountered $(k_1+1)/2$ times. Consider the set $P^{(k_2+1)/2}(R)$. By construction, if there exists a state $X^* = (x^*, 0, \dot{x}^*, \dot{y}^*)$ such that $\dot{x} = 0$, then $X^*$ is an initial condition for a $(k_1, k_2)$ cycler.

Thus, the following approach may be followed. Evolve the unstable manifold of the $L_1$ Lyapunov orbit until $A$ is encountered $(k_1+1)/2$ times and utilize the symmetry $(x, y, \dot{x}, \dot{y}) \mapsto (x, -y, -\dot{x}, \dot{y})$ to obtain the corresponding stable manifold. If there is a well-defined overlap region between the unstable and stable manifolds, then select an interval along the $x$-axis between coordinates $x_l$ and $x_r$ and compute the corresponding $\dot{y}$ values for every $x \in (x_l, x_r)$ from $x$ and $C_0$, thereby constructing the set of candidate cyclers $R$. If an overlap region does not exist, $C_0$ is too large and transit between the two realms is not possible in which case $C_0$ must be reduced. Additionally, if $C_0$ is too small, the region may be not well-defined and it may be difficult to find a sufficient $R$. Once $R$ has been successfully constructed, it is evolved until $B$ is encountered exactly $(k_2+1)/2$ times, producing the set $B$. If $P_B^{(k_2+1)/2}(R)$ does not intersect the $x$-axis, then no $(k_1, k_2)$ cyclers exist at that those values for $\mu$ and $C_0$ and $C_0$ must be reduced further. If $P_B^{(k+1)/2}(R)$ does intersect the $x-$axis, then the value $x^*$ at which $\dot{x} = 0$ can be found via gradient-based methods. As stated above, $x^*$ corresponds to the $x-$coordinate of a cycler orbit, and the full state can be constructed from $x^*$ and $C_0$.

Continuation Methods

Once a $(k_1, k_2)$ cycler is found, nearby cyclers in the same family can be found trivially through continuation methods. Two single-parameter based methods were employed in the present study. In the first method, $C_0$ is incremented by a small $\delta C$ and held constant, and $x_0$ is varied in order to locate the new periodic orbit. Conversely, the second method involves incrementing $x_0$ by a small $\delta x_0$ and varying $C_0$ until a new periodic orbit is found. If the orbit is assumed to be symmetric, the condition that $\dot{x} = 0$ for a periodic orbit can be used as a targeting condition.

## RESULTS

In this section, we give several results on the behavior of symmetric cycler orbit families. In addition, we show some new cycler families found for $\mu$ values in the binary star system range ($0.1 \leq \mu \leq 0.5$).

Earth-Moon Cyclers

The most extensive studies of multi-body cyclers has been in the Earth-Moon system for the purpose of finding efficient pathways to the Moon and key regions of cislunar space. The $(1, 1)$ and $(3, 3)$ cyclers, depicted in Figure 2 in have been of particular interest. The $(1, 1)$ cycler has periods close to a rational multiple of the Earth-Moon synodic period, meaning that these trajectories will likely exist in the elliptic model of the CR3BP. The $(3, 3)$ cycler has 5 stable subfamilies, making it the cycler family with the most stable subfamilies considered thus far. Both orbits are believed to be instrumental in payload distribution for future cislunar infrastructure development.

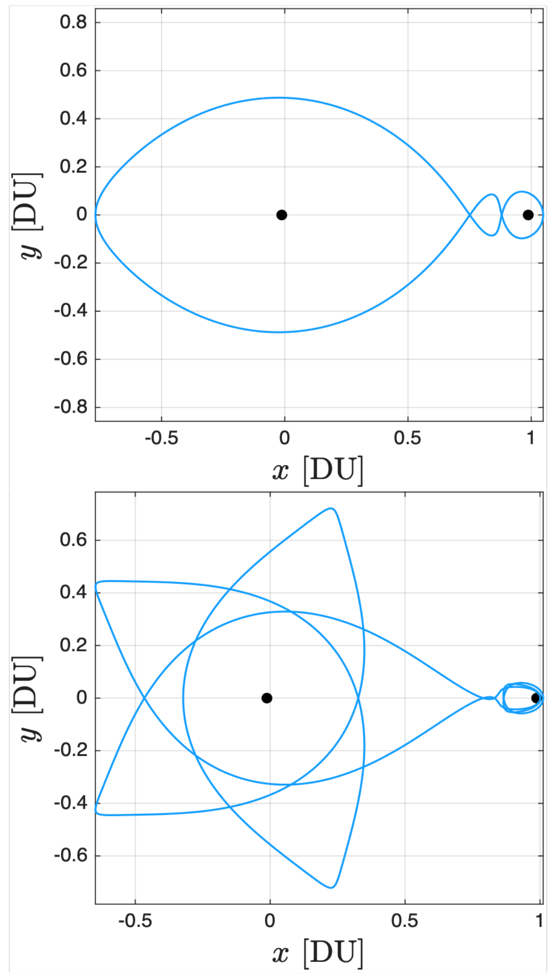


Figure 2: The $(1, 1)$ (above) and $(3, 3)$ (below Earth-Moon cyclers. Both orbits depicted are stable and occur near the saddle-center bifurcation of their respective families.

Binary Star Cyclers

With the discovery that stable sub-families of cycler orbits exist, we must now consider the possibility that planets may exist along cycler orbits in binary star systems. Thus an exploration of cycler orbits for larger $\mu$ values was

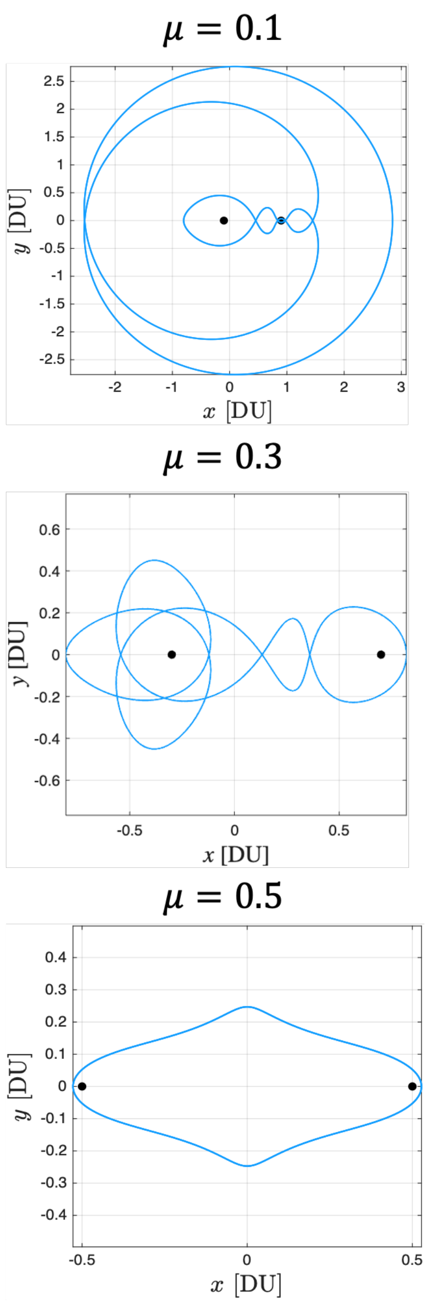


Figure 3: Cycler orbit families found for $\mu$ values in the binary star system range. The top cycler found at $\mu = 0.1$ is of class $(1, 3)$ and is also an example of an exterior cycler, having its second perpendicular crossing in the exterior realm. The middle cycler is a more traditional $(3, 1)$ cycler, found at $\mu = 0.3$. The bottom cycler is a $(1, 1)$ cycler for the special case of equal masses. All cyclers depicted are stable.

conducted. The results are shown in Figure 3.

<u>Theoretical Results.</u>

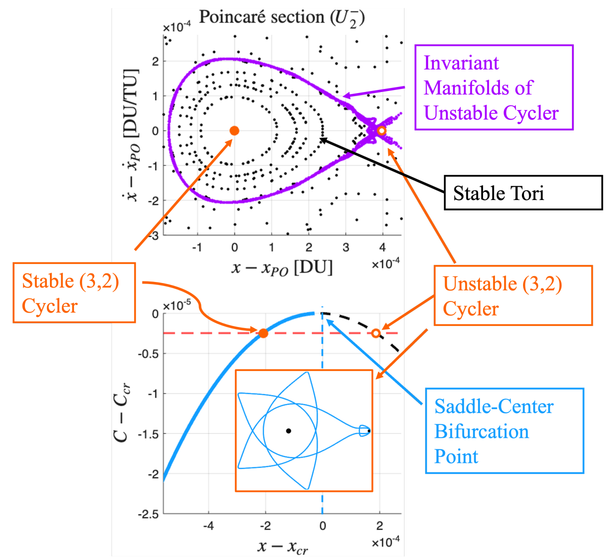


Figure 4: Saddle-Center bifurcation illustration for the (3,2) Earth-Moon cycler family. The top plot shows a Poincaré section containing points near the stable and unstable periodic orbit branches at a Jacobi constant slightly below the critical Jacobi Constant at which the bifurcation occurs. The lower plot shows the Jacobi Constant vs initial $x$ coordinate, which is a common way to illustrate the family. The rotating frame plot of the unstable cycler at the Jacobi Constant of interest is also shown.

With the discovery of stable subfamilies in (cite), the question of whether or not a stable subfamily exists for every $(k_1, k_2)$ cycler family was posed. It is believed at this time that this proposition is true: for every cycler family, there exists at least one stable subfamily.

In all symmetric cycler-families observed, there exists a maximum value of $C$ for which the family exists. At this maximum value of $C$, a saddle-center bifurcation[?] (cite), depicted in Figure 4, occurs. The saddle-center bifurcation, always results in two branches of periodic orbits near the bifurcation point - a stable branch and an unstable branch. Thus if all cycler families undergo the saddle-center bifurcation, then there will always exist one stable sub-family near the bifurcation.

It has not yet been rigorously proven that all symmetric cycler families undergo a saddle-center bifurcation, but it is believed to be highly probable at this stage. Since transit between the primary and secondary realms is impossible for large enough $C$, and since the Global Orbit Theorem[4] guarantees the existence of unstable $(k_1, k_2)$ cyclers for small enough $C$, there must exist a critical point $C^*$ in between, where the family is ”created”. The only bifurcation that has been observed in this way is the Saddle-Center type.

An additional conjecture on the nature of symmetric cyclers was stated earlier: at a given $\mu$, and for any $C$, there exist at minimum no symmetric cyclers, and at maximum, two symmetric cyclers. The evidence for this result is based on the geometric method described in an earlier section. The image $P_B^{k_2+1/2}(R)$ is such that only two intersections with $\dot{x} = 0$ exist for a given $R$. It is believed that this, too, can be proven based on the mathematical properties of the CR3BP, although the strategy is less clear in this case.

## CONCLUSIONS

Several new cycler trajectories for values of $\mu$ in the binary star mass parameter range were found in the present study, including an exterior cycler for $\mu = 0.1$, and a cycler in the equal mass case of the CR3BP ($\mu = 0.1$). In addition, substantial evidence was found in favor of the existence of at least one stable subfamily for every symmetric $(k_1, k_2)$ cycler in the PCR3BP.

There are several unanswered questions that remain with respect to the multi-orbiter cyclers. We have yet to provide a rigorous argument to prove the existence of a stable subfamily for every cycler family. Additionally, it is unknown whether these results would continue to hold in the spacial problem. Another question is whether or not some cyclers can persist with significant perturbations such as the inclusion of eccentricity or a fourth body. Finally, while symmetric cyclers have been studied extensively, it is known that asymmetric cyclers exist, but due to their numerical complexity, much less is known about these families and it is not so clear that the same results will hold.

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