INFLATION OF STRAIN-STIFFENING RUBBER-LIKE THIN SPHERICAL SHELLS

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Abstract

In this paper, we investigate the predictions of several widely used strain-stiffening phenomenological constitutive models for the classical limit point instability that is well-known to occur in the inflation of internally pressurized rubber-like spherical thin shells (balloons). The shells are composed of incompressible isotropic nonlinearly elastic materials. For a variety of specific strain-energy densities that give rise to strain-stiffening in the stress-stretch response, the inflation pressure versus stretch relations are given explicitly and the monotonicity, or lack thereof, of the inflation curves is examined. While such results are known for constitutive models that exhibit a gradual stiffening (e.g. exponential and power-law models), our primary focus is on materials that undergo *severe* strain-stiffening in the stress-stretch response. In particular, we consider two recently developed constitutive models that reflect limiting chain extensibility at the molecular level. Results for classical models are also presented for comparison. It is shown that for materials with *sufficiently low extensibility* no limit point instability occurs and so *stable inflation* is then predicted for such materials. The results have a variety of aerospace applications, e.g. to deployable space structures and NASA Weather Balloons.

1. Introduction

It is well-known that elastic rubber-like materials undergoing large deformations can exhibit a variety of interesting instabilities. A review of some of the more intriguing examples has been given in a recent paper by Gent [12]. In particular, the work [12] provides a convincing demonstration of the predictive capability of the theory of hyperelasticity based on phenomenological constitutive models involving strain-energy densities to model the onset of such instabilities. The concern of this paper is with one particular problem considered in [11, 12] for *isotropic incompressible* hyperelastic materials, namely the inflation of internally pressurized spherical rubber thin shells (balloons). This classical problem has been widely investigated within the theory of finite hyperelasticity (see, e.g. [15], [32]) largely motivated by applications to rubber. It has also been recognized that the instabilities arising in such inflation problems are of considerable interest in the context of biological materials. Our particular interest here is on further examination of these instabilities for special classes of constitutive models that give rise to severe *strain-stiffening* in their stress response curves at large strains. The constitutive models that we employ reflect *limiting chain extensibility* at the molecular level and thus are appropriate for modeling non-crystallizing elastomers, such as rubbery and biological materials.

In the next Section, we discuss some preliminaries from the theory of nonlinear hyperelasticity for isotropic incompressible solids. In particular, we describe some phenomenological constitutive models that exhibit strain-stiffening at large strains. The first class of models reflects limiting chain extensibility at the molecular level and gives rise to *severe* strain-stiffening in the stress-stretch response. The second class exhibits a less abrupt strain-stiffening. Examples are the exponential models widely used in biomechanics and power-law models. In Section 3, we summarize results for the problem of inflation of a spherical balloon. This problem has been extensively investigated in the literature on nonlinear elasticity, but here we extend the discussion to include contemporary models that demonstrate severe strain-stiffening. The problem of spherical inflation is interesting as it provides a simple context in which to demonstrate the appearance of a *limit point instability*, i.e., a point in the inflation curve (inflation pressure versus stretch relation) where a local maximum in pressure occurs after which the volume increases with decreasing pressure. It is well known that the character of such an instability depends crucially on the constitutive model employed. For example, it is well known that a balloon composed of a neo-Hookean material *always* has an inflation curve with a single local pressure maximum with subsequent monotonic decreasing pressure. Moreover, the stretch at which the pressure maximum occurs is *quasi-universal*, i.e., is the same for *all* neo-Hookean materials. However, in the case of a Mooney-Rivlin or exponential constitutive model, for a certain range of the material parameters the inflation curve can have a local maximum followed by a local minimum after which the pressure increases again until the balloon bursts. This predicted behavior is, of course, in agreement with experimental observation.

Our primary concern in this paper is to investigate the occurrence and character of turning points in the inflation curve for limiting chain extensibility models. It is shown in Section 3 that for each of two particular such models, both of which involve a single limiting chain extensibility parameter, there is a *minimum* value of this parameter below which no limit point instability occurs. *Thus, for rubber-like materials of sufficiently low extensibility, a stable inflation is predicted*. This result is consistent with observations made in the context of aneurysms in biological tissues (see, e.g., [25-27]) where exponential models are used and is of importance in view of the demonstrated potential of the application of limiting chain extensibility models in biomechanics (see, e.g., [20, 16]). We refer to Humphrey [25, 26] for an extensive discussion of the applications of nonlinear elasticity theory to biomechanics.

2. Preliminaries

In continuum mechanics, the mechanical properties of elastomeric materials are described in terms of a strain-energy density function W. If the left Cauchy-Green tensor is denoted by $\mathbf{B} = \mathbf{F}\mathbf{F}^{\mathrm{T}}$, where $\mathbf{F}$ is the gradient of the deformation and $\lambda_1, \lambda_2, \lambda_3$ are the principal stretches, then, for an isotropic material, W is a function of the strain invariants

$$I_1 = tr\mathbf{B} = \lambda_1^2 + \lambda_2^2 + \lambda_3^2,$$
$$I_2 = \frac{1}{2}\left[\left(tr\mathbf{B} \right)^2 - tr\left(\mathbf{B}^2 \right) \right] = \lambda_1^2\lambda_2^2 + \lambda_2^2\lambda_3^2 + \lambda_3^2\lambda_1^2,$$
$$I_3 = \det \mathbf{B} = \lambda_1^2\lambda_2^2\lambda_3^2. \tag{2.1}$$

Rubber-like materials are often assumed to be *incompressible* provided that the hydrostatic stress does not become too large and so the admissible deformations must be isochoric, i.e., $\det \mathbf{F} = 1$ so that $I_3 = 1$. The response of an incompressible isotropic elastic material can be determined by applying the standard constitutive law (see, e.g., Ogden [32], Beatty [3])

$$\mathbf{T} = -p\mathbf{1} + 2\frac{\partial W}{\partial I_1}\mathbf{B} - 2\frac{\partial W}{\partial I_2}\mathbf{B}^{-1}, \tag{2.2}$$

where p is a hydrostatic pressure term associated with the incompressibility constraint and $\mathbf{T}$ denotes the Cauchy stress.

The classical strain-energy density for *incompressible* rubber is the Mooney-Rivlin strain-energy

$$W = \frac{1}{2}\mu\left[\alpha(I_1 - 3) + (1-\alpha)\left(I_2 - 3 \right) \right], \tag{2.3}$$

where $\mu > 0$ is the constant shear modulus for infinitesimal deformations and

$0 < \alpha \leq 1$ is a dimensionless constant. When $\alpha = 1$ in (2.3), one obtains the neo-Hookean strain-energy

$$W = \frac{\mu}{2}(I_1 - 3), \qquad (2.4)$$

which corresponds to a Gaussian statistical mechanics model, and is often referred to as the kinetic theory of rubber. While the Mooney-Rivlin and neo-Hookean models are useful in describing rubber-like materials at small stretches, the theoretical predictions based on (2.3) and (2.4) do not adequately describe experimental data for rubber at high values of strain. For example, neither strain-energy is able to describe the characteristic S-shaped load versus stretch curve exhibited in simple tension experiments. For modeling of soft biological tissue, where *rapid* strain stiffening occurs even at moderate stretches (see, e.g. [16, 25, 26]), classical models are completely inappropriate. To model such *stiffening,* a number of alternative models have been proposed. In the molecular theory of elasticity (see, e.g., [8] for a recent review) these models are usually called non-Gaussian, because they introduce a distribution function for the end-to-end distance of the polymeric chain composing the rubber-like material which is not Gaussian. Such models are applicable to many other materials such as low-density polyethylenes [13], wool and hair fibers [14], and DNA molecules [29]. From the phenomenological point of view, the non-Gaussian models of concern here can be divided into two classes: models with limiting chain extensibility and strain-hardening models. *Our emphasis is primarily on the former.*

One such class of *isotropic* incompressible models with a maximum achievable length of the polymeric molecular chains composing the material is described by strain-energies of the form $W(I_1, I_2, I^*)$ where I^* is a limiting chain extensibility parameter. The function W is such that the stress components are unbounded as $f(\lambda_1, \lambda_2, \lambda_3) \to I^*$, where f is some prescribed function, and so one must impose the *constraint*

$$f(\lambda_1, \lambda_2 \lambda_3) < I^*, \qquad (2.5)$$

on admissible deformations. One such model is a *three-parameter* model due to Gent [10-12], who proposed the strain-energy density

$$W_I = \frac{\mu}{2}\left[-\alpha J_m \, ln\left(1 - \frac{I_1 - 3}{J_m}\right) + (1-\alpha)(I_2 - 3)\right], \qquad (2.6)$$

where μ is the shear modulus for infinitesimal deformations, α $(0 < \alpha \leq 1)$ is a dimensionless constant and J_m is the limiting chain extensibility parameter. It is easily shown that each component of $\mathbf{T} \to \infty$ as $I_1 - 3 \to J_m$, so that $f(\lambda_1, \lambda_2, \lambda_3) = I_1 - 3$ for this model. On taking the limit as $J_m \to \infty$ in (2.6) we recover the Mooney-Rivlin model (2.3). Other related three-parameter models with more elaborate dependence on I_2 are discussed in [12, 33, 34].

For the case when $\alpha = 1$ in (2.6), first proposed by Gent [10] in 1996, we obtain the *two-parameter* generalized neo-Hookean model (i.e., $W = W(I_1)$)

$$W_G = -\frac{\mu}{2} J_m \, ln\left(1 - \frac{I_1 - 3}{J_m}\right), \qquad (2.7)$$

$$I_1 < J_m + 3,$$

(henceforth called *the basic Gent model*), and one recovers the neo-Hookean model on taking the limit as $J_m \to \infty$ in (2.7). The stress response for (2.7) in simple extension is described in [10] and elastic instabilities of inflated rubber shells have been examined in [11, 12]. For rubber, typical values for the dimensionless parameter J_m for simple extension range from 30-100 whereas for biological tissue, much smaller values of J_m are appropriate. For example, for human arterial wall tissue, values on the order of 0.4 to 2.3 have been suggested [20]. The basic Gent model (2.7) gives theoretical predictions similar to the more complicated Arruda and Boyce eight chain model based on inverse Langevin function compact support statistics (see, e.g., [2, 4, 5]). A molecular basis for the basic Gent model was given in [19]. It is shown that the infinitesimal shear modulus is $\mu = nkT$ as is usual in the molecular models, where k is the Boltzmann constant, T is the absolute temperature, and n is the chain density, while $J_m = 3(N-1)$, where N is the number of links in a single chain. Thus the basic Gent phenomenological model predicts similar behavior to the molecular models, has a clear microscopic interpretation for the constitutive coefficients and is tractable analytically. On using (2.2), we find that the Cauchy stress associated with (2.7) is given by

$$\mathbf{T} = -p\mathbf{I} + \mu \frac{J_m}{J_m - (I_1 - 3)}\mathbf{B}, \qquad (2.8)$$

so that the stress has a singularity as $I_1 \to J_m + 3$, reflecting the rapid strain stiffening observed in experiments.

An alternative *two-parameter* limiting chain extensibility model with $W(I_1, I_2, I^*)$ has been proposed recently in [21] where

$$W_N = -\frac{\mu}{2}\frac{(J-1)^2}{\mathrm{J}}\, ln\left(\frac{1}{(J-1)^3}\left(J^3 - J^2 I_1 + J I_2 - 1\right)\right),$$

$$J I_1 - I_2 < \frac{J^3 - 1}{J}, \quad J > 1, \qquad (2.9)$$

or, on using the principal stretches of the deformation,

$$W_N = -\frac{\mu}{2}\frac{(J-1)^2}{\mathrm{J}}\ln\left(\frac{\left(1-\frac{\lambda_1^2}{J}\right)\left(1-\frac{\lambda_2^2}{J}\right)\left(1-\frac{\lambda_3^2}{J}\right)}{\left(1-\frac{1}{J}\right)^3}\right),$$

$$\lambda_1\lambda_2\lambda_3 = 1. \qquad (2.10)$$

In (2.9), (2.10), μ is the shear modulus for infinitesimal deformations. Note that the definitions of W_N here differ from those in [21-24] by a factor of $(J-1)^2/J^2$. The limiting chain extensibility parameter J is the square of the maximum stretch allowed by the finite extensibility of the chains so that

$$f(\lambda_1, \lambda_2, \lambda_3) = \max\left(\lambda_1^2, \lambda_2^2, \lambda_3^2\right) < J. \qquad (2.11)$$

Again, in the limit as $J \to \infty$ in (2.9) or (2.10), we recover the neo-Hookean model (2.4). It is important to point out the difference between the *constraint* (2.11) and the *constraint* $I_1 < J_m + 3$ arising in connection with the Gent model. As already pointed out in [18], [21], the limiting chain condition expressed in terms of the principal invariant is less physically accessible than (2.11). Furthermore, the absence of the dependence on the *second* invariant in the basic Gent model entails some physical limitations. Thus the W_N model has advantages over the basic Gent model. Note that (2.10) belongs to the class of models for which $W(\lambda_1, \lambda_2, \lambda_3, J)$, with $\lambda_1\lambda_2\lambda_3 = 1$ because of incompressibility. For such models, the limiting chain extensibility constraint is given in terms of the principal stretches directly and this has some advantages from a physical point of view.

While our primary concern here is with limiting chain extensibility models such as the above that exhibit *severe* strain-stiffening, we note that there are numerous strain-hardening constitutive models that have been successfully employed to investigate the effects of a *less abrupt* strain-stiffening. A generalized neo-Hookean model of this type widely used in the biomechanics literature is the two-parameter strain-energy density

$$W = \frac{\mu}{2b}\left\{\exp\left[b\left(I_1 - 3\right)\right] - 1\right\}, \qquad (2.12)$$

where the dimensionless constant $b > 0$. This *exponential* model was first proposed by Fung [9]. On taking the limit as $b \to 0$ in (2.12) we again recover the neo-Hookean model (2.4). Another *strain-stiffening* generalized neo-Hookean model that has been extensively employed in numerous applications is the three-parameter Knowles' power-law model [28]

$$W = \frac{\mu}{2b}\left[\left(1+\frac{b}{n}(I_1-3)\right)^n - 1\right], \qquad (2.13)$$

where $b > 0$, $n > 1$. When $n = 1$ in (2.13), one recovers the neo-Hookean model (2.4), while in the limit as $n \to \infty$, (2.13) reduces to (2.12) (see [17]). An alternative *two-parameter* power-law model for which $W = W(I_1, I_2)$ is

$$W = \frac{2\mu}{m^2}(\lambda_1^m + \lambda_2^m + \lambda_3^m - 3)\ , \ \lambda_1\lambda_2\lambda_3 = 1, \quad (2.14)$$

which is strain stiffening if $m > 2$. When $m = 2$ in (2.14), one again recovers the neo-Hookean model. The well-known six-parameter model of Ogden [32] is a generalization of (2.14).

It is worth noting that, while the exponential and power-law models reflect strain hardening, they do *not* exhibit the *rapid strain stiffening* characteristic of the limiting chain extensibility models. This is a notable distinction between them. A recent paper by Chagnon et al. [7] suggests that both types of models are essentially equivalent but as was discussed in [22, 23] there are important physical differences in their predictions. For example, as described in detail in [17], the shear stresses at the tip of a Mode III crack are *singular* for the power-law model (2.13) whereas these crack tip stresses are *bounded* for the Gent model (2.7).

3. Inflation of a Spherical Balloon

We consider the classical problem of inflation of a thin spherical shell with undeformed radius r_o and thickness $t_o << r_o$. On assuming that the balloon remains spherical on inflation, it is well-known that the relation between the inflation pressure (difference between atmospheric and applied pressure) and stretch $\lambda = r/r_0$ for an isotropic incompressible elastic material is

$$P(\lambda) = \frac{4t_0}{r_0}(\lambda^{-1} - \lambda^{-7})\left(\frac{\partial W}{\partial I_1} + \lambda^2\frac{\partial W}{\partial I_2}\right) = \frac{t_0}{r_0\lambda^2}\frac{\partial W}{\partial \lambda}. \qquad (3.1)$$

For this problem, on denoting by λ_1 , λ_2 the principal stretches in the spherical surface and by λ_3 that in the thickness direction, one has $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = \lambda^{-2}$. Since the stretches are constant for a thin shell, we have a state of *equibiaxial extension*. The derivatives in (3.1) are evaluated at $I_1 = 2\lambda^2 + \lambda^{-4}$, $I_2 = \lambda^4 + 2\lambda^{-2}$. (A concise derivation of this result is given by Beatty [3]). The relation between the inflating pressure and the stretch λ given in (3.1) has been examined previously by a number of authors for a variety of strain-energy densities. On defining a normalized dimensionless pressure

$$\overline{p}(\lambda) = \frac{r_0 P(\lambda)}{2\mu t_0}, \qquad (3.2)$$

Beatty [3] shows that (3.1) can be written as

$$\overline{p}(\lambda) = k(\lambda)(\lambda^{-1} - \lambda^{-7}) \qquad (3.3)$$

where

$$k(\lambda) = \begin{cases} 1 & \text{neo-Hookean (2.4)}, \\ 1 + \dfrac{(1-\alpha)}{\alpha}\lambda^2 & \text{Mooney-Rivlin (2.3)}, \\ e^{b(2\lambda^2+\lambda^{-4}-3)} & \text{exponential (2.12).} \end{cases} \qquad (3.4)$$

For the power-law models (2.13) and (2.14) equation (3.1) can be used to show that

$$k(\lambda) = \left[1 + \frac{b}{n}(2\lambda^2 + \lambda^{-4} - 3)\right]^{n-1} \qquad (3.5)$$

and

$$k(\lambda) = \frac{2}{m}\left(\frac{\lambda^{m-3} - \lambda^{-2m-3}}{\lambda^{-1} - \lambda^{-7}}\right), \qquad (3.6)$$

respectively.

For the basic Gent model (2.7), one finds that

$$k(\lambda)=\left[\frac{J_m}{J_m-(2\lambda^2+\lambda^{-4}-3)}\right], \quad (3.7)$$

where the maximum allowable stretch λ_m occurs when $2\lambda_m^2+\lambda_m^{-4}=J_m+3$, which we write as the cubic equation

$$2x^3-(J_m+3)x^2+1=0\ , \qquad (x=\lambda_m^2)\,. \quad (3.8)$$

For the model (2.10), we have

$$k(\lambda)=\left(\frac{J-1}{J}\right)^2\left[\frac{J^2\lambda^4}{(\lambda^4 J-1)(J-\lambda^2)}\right], \quad (3.9)$$

and

$$\lambda_m^2=J\ . \quad (3.10)$$

The variation of the normalized pressure $\bar{p}(\lambda)$ with λ for the various models is readily ascertained. From (3.3) - (3.7) and (3.9), we see that $\bar{p}(\lambda)$ constitutes a one-parameter family of relations except in the special case of the neo-Hookean model where $\bar{p}(\lambda)$ is *independent* of material parameters. In this case, it is well known that $\bar{p}(\lambda)$ is monotonically increasing in λ, has a single maximum at the *universal* stretch (i.e., at the same stretch for *all* neo-Hookean materials)

$$\lambda^*=\sqrt[6]{7}\approx 1.383\ , \quad (3.11)$$

and then decreases monotonically to zero as λ further increases (see, e.g., Beatty [3]). For all the other models considered, one seeks the location of extreme points by setting

$$\frac{d\bar{p}}{d\lambda}=0\,. \quad (3.12)$$

For the second and third of (3.4), this analysis has been carried out by Beatty [3]. It is shown in [3] that for the Mooney-Rivlin model, where $k(\lambda)$ is given by $(3.4)_2$, equation (3.12) yields

$$\frac{(1-\alpha)}{\alpha}\lambda^8-\lambda^6+\frac{5(1-\alpha)}{\alpha}\lambda^2+7=0, \quad (3.13)$$

while for the exponential model (2.15), one finds that

$$4b\lambda^{12}-\lambda^{10}-8b\lambda^6+7\lambda^4+4b=0\,. \quad (3.14)$$

In the limit as $\alpha\to 1$ in (3.13) or $b\to 0$ in (3.14), one recovers the result (3.11) for the neo-Hookean model. As discussed in detail by Beatty [3] (see also Section 4.4.4 of Humphrey [25]), equations (3.13) and (3.14) have at most *two* positive real roots that depend on the material parameters. It is also shown in [3, 25] that for $0.824<\alpha<1$, the pressure for the Mooney-Rivlin model rises to a maximum, decreases to a minimum, and then increases to infinity with increasing stretch. For $\alpha<0.824$, (3.13) has no real roots so that $d\bar{p}/d\lambda>0$ for all λ and the inflation curve is simply monotonic increasing. It is shown in Beatty [3] that similar results hold for the exponential model (2.12). In this case, for very soft tissue for which $0 < b < 0.067$, one obtains a pressure maximum and minimum behavior similar to that for the Mooney-Rivlin material except that the inflation curves rise more steeply for large stretch for the exponential model. For tissue that stiffens more rapidly for which $b > 0.067$, the inflation curve is *monotonic increasing*. See also the discussion on p. 406 of [25] and on p. 57 of [27] where no limit point instability is predicted for a specific biological tissue modeled by an (anisotropic) exponential model. For the power-law model (2.14), the corresponding ranges for m are $2 < m < 3$ and $m > 3$ respectively (see, e.g., Ogden [31] or Holzapfel [15], p. 249).

We turn now to the limiting chain extensibility models of main concern. For the Gent model (2.7), we find from (3.3), (3.7) that (3.12) holds if

$$6\lambda^{12}-(J_m+3)\lambda^{10}-21\lambda^6+7(J_m+3)\lambda^4-3=0\ . \quad (3.15)$$

In the limit as $J_m\to\infty$, one recovers the neo-Hookean result (3.11). If $J_m>J_{mt}\equiv 17.6$, then it can be shown that (3.15) has *two* real roots $\lambda > 1$ consistent with (3.8) and one obtains an inflation behavior somewhat similar to that described above for the Mooney-Rivlin and exponential models in that the inflation curve rises to a maximum, decreases to a minimum, and then increases to infinity. The normalized pressure $\overline{p}$ defined in (3.2) is now written as $\overline{p}(\lambda, J_m)$ to emphasize the dependence on the limiting chain parameter J_m. Note that $\overline{p}(\lambda, J_m) \to \infty$ at a *finite* value of λ for each J_m, given by the root of (3.8), reflecting limiting chain extensibility. *Thus, for large stretch, the inflation curve for the Gent model rises much more rapidly than that for the Mooney-Rivlin and exponential models and this behavior is in closer agreement with experiments (see, e.g., Beatty* [3] *for a discussion of experimental data*). Results similar to the foregoing were also obtained in Gent [11, 12] and by Pucci and Saccomandi [35]. *However, if $J_m < J_{mt}$, then there are no extrema.* For arterial wall tissue, it has been shown in [20] that values of J_m can be expected to be of order unity and so *no instability is predicted for the Gent model in spherical inflation of such biological tissues.* In Fig. 1, we plot the normalized pressure versus stretch for three representative values of J_m, i.e., J_m = 30, 17.6, and 5 for which the corresponding values of the limiting stretch $_m$, given by (3.8), are $_m$ = 4.06, 3.21, and 1.99 respectively. When J_m = 30, the inflation curve has a maximum at $\lambda^* = 1.45$, which is only slightly larger than the value (3.11) for the neo-Hookean model (cf. Gent [11, 12]).

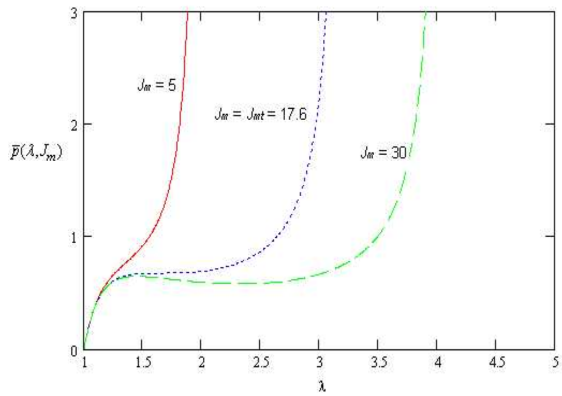

Fig. 1. Pressure vs. stretch for the Gent model in balloon inflation with J_m = 30, 17.6, 5.

For the W_N model, we find from (3.3), (3.9) that (3.12) holds if

$$3J\lambda^{12} - J^2\lambda^{10} + \lambda^8 - 12J\lambda^6 + 7J^2\lambda^4 + 5\lambda^2 - 3J = 0 . \quad (3.16)$$

Again, on taking the limit as $J \to \infty$ in (3.16), we recover the neo-Hookean result (3.11). As for the Gent model, there exists a *transition value* $J_t = 10.1$ such that for $J > J_t$, (3.16) has *two* real roots in the range of interest corresponding to the inflating pressure reaching a local maximum, a decrease to a minimum, and then an increase to infinity at a finite value of stretch. For $J < J_t$, however, (3.16) has no real roots and the pressure monotonically increases to infinity. These results are illustrated in Fig. 2, where the normalized pressure $\overline{p}(\lambda, J)$ versus stretch is plotted for three representative values of J, i.e., J = 18, 10.1, and 5 corresponding to which the limiting values of stretch, given by (3.10), are $_m$ = 4.24, 3.18 and 2.24, respectively. Observe that the limiting values of stretch $_m$ corresponding to J_{mt} and J_t are virtually the same so that the transition values of the limiting chain parameters for both models *correspond to materials with almost the same ultimate extensibility*. For J = 18, the inflation curve has a maximum at $\lambda^* = 1.43$.

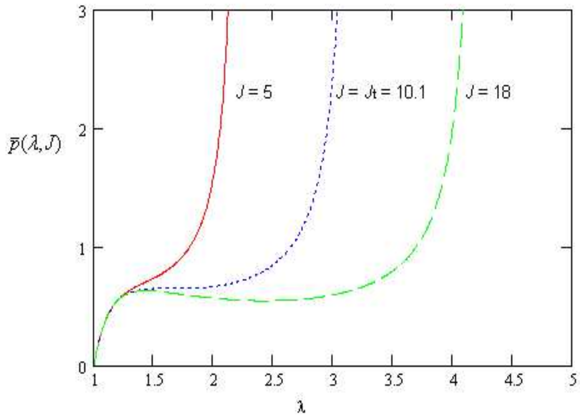

Fig. 2. Pressure vs. stretch for the W_N model in balloon inflation with J = 18, 10.1, 5.

It is also of interest to consider the membrane stresses in the inflated balloon. Since we have a state of equibiaxial extension, one finds that the Cauchy stresses are $T_1 = T_2 = T$ given by

$$T = 2\left(\lambda^2 - \lambda^{-4}\right)\left(\frac{\partial W}{\partial I_1} + \lambda^2 \frac{\partial W}{\partial I_2}\right). \quad (3.17)$$

The non-dimensional Cauchy stresses for the Gent and the W_N models are thus

$$\bar{T} = \frac{T}{\mu} = (\lambda^2 - \lambda^{-4}) \frac{J_m}{J_m - (2\lambda^2 + \lambda^{-4} - 3)} \quad (3.18)$$

and

$$\bar{T} = \frac{T}{\mu} = \left(J - 1\right)^2 \frac{\lambda^6 - 1}{(\lambda^4 J - 1)(J - \lambda^2)} \quad (3.19)$$

respectively. For a *given* J_m, the maximum allowable stretch λ_m is computed as a root of (3.8), and then, from (3.10), the corresponding value of J is given by $J = \lambda_m^2$. For example, when $J_m = 30$, we have shown earlier that $\lambda_m = 4.06$ and thus the corresponding value of J is $J = 16.498$. The normalized Cauchy stress response corresponding to these values is shown in Fig. 3 where it is seen that the response of both models is virtually identical. The corresponding curves for specific Mooney-Rivlin and Ogden models (see, for example, Holzapfel [15], Figure 6.3) also exhibit strain-stiffening but not as severe as that shown in Fig. 3.

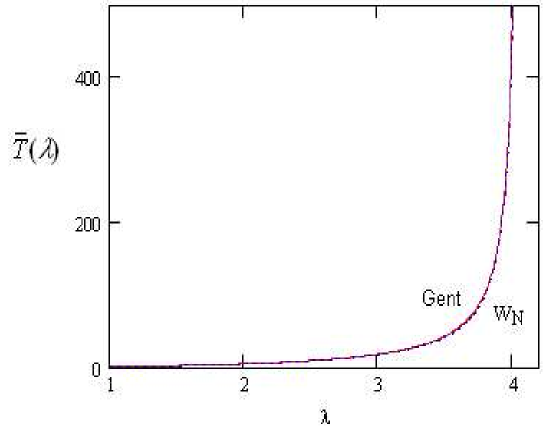

Fig. 3. Cauchy stress response for the Gent and W_N models in balloon inflation (J_m = 30, J = 16.498).

Finally, we remark that several aspects of the balloon inflation problem have been investigated by many authors for *general* incompressible material models. See, e.g., Ogden [31] for further discussion. A general treatment of pressure maxima for inflation of thin and thick-walled internally pressurized hollow spheres has been given by Carroll [6]. The results obtained earlier in this section may also be obtained on specialization of the foregoing general considerations, but we consider it instructive from a physical standpoint to present explicit results for widely used specific strain-energies as in the preceding. We also remark that it is well-known that loss of spherical symmetry may occur in balloon inflation after the pressure maximum is attained, but such concerns are beyond the scope of the present work (see, e.g., Alexander [1], Needleman [30], Ogden [31], and the references cited therein).

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